

Hamiltonian structure of the complex Monge-Ampère equation

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Abstract

We discover Hamiltonian structure of the complex Monge-Ampère equation when written in a first order two-component form. We present Lagrangian and Hamiltonian functions, a symplectic form and the Hamiltonian operator that determines the Poisson bracket.

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1 Introduction

In earlier papers [1,2] we presented complex multi-Hamiltonian structure of Plebański's second heavenly equation [3], which by Magri's theorem [4] proves that it is a completely integrable system in four complex dimensions. A first important step for obtaining this result was the discovery of a Hamiltonian structure of the second heavenly equation that was set into a two-component evolutionary form. In this paper our goal was to discover a Hamiltonian structure of the the complex Monge-Ampère equation (*CMA*) as a first step to obtaining its multi-Hamiltonian representation and hence its complete integrability in the sense of Magri.

This paper is based on our unpublished results obtained two years ago where we overlooked that, in fact, we had then obtained a symplectic, and hence Hamiltonian, structure of *CMA*.

In section 2 we set the complex Monge-Ampère equation in a two-component first-order evolutionary form and find the form of its Lagrangian that is appropriate for Hamiltonian formulation. In section 3 we discover a symplectic structure and Hamiltonian structure of this *CMA* system.

2 Complex Monge-Ampère equation in first-order evolutionary form and its Lagrangian

The complex Monge-Ampère equation has the form

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = \varepsilon \quad (2.1)$$

where u is a real-valued function of the two complex variables z^1, z^2 and their conjugates \bar{z}^1, \bar{z}^2 , the subscripts denote partial derivatives with respect to these variables. Here ε is an arbitrary constant which means in essence that either $\varepsilon = \pm 1$ or $\varepsilon = 0$, though we shall not be interested here in the latter simple special case.

CMA (2.1) is a second-order partial differential equation, so in order to discuss its Hamiltonian structure, we shall single out a real independent variable, $t = 2\Re z^1$, in (2.1) to play the role of “time”, introduce its companion real variable $x = 2\Im z^1$ and change the notation for the second complex variable $z^2 = w$. Then (2.1) becomes

$$(u_{tt} + u_{xx})u_{w\bar{w}} - u_{tw}u_{t\bar{w}} - u_{xw}u_{x\bar{w}} + i(u_{tw}u_{x\bar{w}} - u_{xw}u_{t\bar{w}}) = \varepsilon. \quad (2.2)$$

Now we can express (2.2) as a pair of first-order nonlinear evolution equations by introducing an auxiliary unknown $v = u_t$

$$\begin{cases} u_t = v \\ v_t = -u_{xx} + \frac{1}{u_{w\bar{w}}} (v_w v_{\bar{w}} + u_{xw} u_{x\bar{w}} + i(v_{\bar{w}} u_{xw} - v_w u_{x\bar{w}}) + \varepsilon), \end{cases} \quad (2.3)$$

so that finally (2.1) adopts a two-component form. For the sake of brevity we shall henceforth refer to (2.3) as CMA system.

The Lagrangian density for the original form (2.1) of the complex Monge-Ampère equation was suggested in [5]

$$L = \frac{1}{6} [u_1 u_{\bar{1}} u_{2\bar{2}} + u_2 u_{\bar{2}} u_{1\bar{1}} - u_1 u_{\bar{2}} u_{2\bar{1}} - u_2 u_{\bar{1}} u_{1\bar{2}}] + \varepsilon u, \quad (2.4)$$

which in our new notation for independent variables becomes

$$\begin{aligned} L = \frac{1}{6} [(u_t^2 + u_x^2)u_{w\bar{w}} + u_w u_{\bar{w}}(u_{tt} + u_{xx}) \\ - u_{\bar{w}}(u_t - iu_x)(u_{tw} + iu_{xw}) - u_w(u_t + iu_x)(u_{t\bar{w}} - iu_{x\bar{w}})] + \varepsilon u. \end{aligned} \quad (2.5)$$

For our purposes, we choose the Lagrangian density for the first-order *CMA* system (2.3) to be linear in the time derivatives of the unknowns u_t and v_t :

$$L = \frac{1}{6} \{ (4vu_t - 3v^2 + u_x^2)u_{w\bar{w}} + u_w u_{\bar{w}}(v_t + u_{xx}) - u_x(u_{\bar{w}}u_{xw} + u_w u_{x\bar{w}}) \\ - u_t(u_{\bar{w}}(v_w + 2iu_{xw}) + u_w(v_{\bar{w}} - 2iu_{x\bar{w}})) \} + \varepsilon u \quad (2.6)$$

which, after substituting $v = u_t$, coincides with our original Lagrangian (2.5) up to a total divergence.

3 Symplectic two-form and first Hamiltonian structure

Since the Lagrangian density (2.6) is linear in u_t and v_t , the canonical momenta

$$\pi_u = \frac{\partial L}{\partial u_t} = \frac{1}{6} [4vu_{w\bar{w}} - u_{\bar{w}}(v_w + 2iu_{xw}) - u_w(v_{\bar{w}} - 2iu_{x\bar{w}})] \\ \pi_v = \frac{\partial L}{\partial v_t} = \frac{1}{6} u_w u_{\bar{w}} \quad (3.1)$$

cannot be inverted for the velocities u_t and v_t and the Lagrangian is thus degenerate. Therefore, following Dirac [6], we impose them as constraints

$$\phi_u = \pi_u + \frac{1}{6} [-4vu_{w\bar{w}} + u_{\bar{w}}(v_w + 2iu_{xw}) + u_w(v_{\bar{w}} - 2iu_{x\bar{w}})] = 0 \\ \phi_v = \pi_v - \frac{1}{6} u_w u_{\bar{w}} = 0 \quad (3.2)$$

and calculate the Poisson brackets of the constraints (more details of the procedure were given in [1])

$$K_{ik} = [\phi_i(x, w, \bar{w}), \phi_k(x', w', \bar{w}')] \quad (3.3)$$

organizing them in a matrix form. This yields us the inverse of the Hamiltonian operator

$$K = \begin{pmatrix} (v_{\bar{w}} - iu_{x\bar{w}})D_w + (v_w + iu_{xw})D_{\bar{w}} + v_{w\bar{w}} & -u_{w\bar{w}} \\ u_{w\bar{w}} & 0 \end{pmatrix} \quad (3.4)$$

as an explicitly skew-symmetric local operator. A symplectic 2-form is a volume integral $\Omega = \int_V \omega dx dw d\bar{w}$ of the density

$$\omega = \frac{1}{2} du^i \wedge K_{ij} du^j = \frac{1}{2} (v_{\bar{w}} - i u_{x\bar{w}}) du \wedge du_w + \frac{1}{2} (v_w + i u_{xw}) du \wedge du_{\bar{w}} + u_{w\bar{w}} dv \wedge du \quad (3.5)$$

where $u^1 = u$ and $u^2 = v$. In ω , under the sign of the volume integral, we can neglect all the terms that are either total derivatives or total divergencies due to suitable boundary conditions on the boundary surface of the volume.

For the exterior differential of this 2-form we obtain

$$\begin{aligned} d\omega = -i du_x \wedge du_w \wedge du_{\bar{w}} = -(i/3) & \left(D_x (du \wedge du_w \wedge du_{\bar{w}}) \right. \\ & \left. + D_w (du_x \wedge du \wedge du_{\bar{w}}) + D_{\bar{w}} (du_x \wedge du_w \wedge du) \right) \iff 0 \end{aligned} \quad (3.6)$$

that is, a total divergence which is equivalent to zero, so that the 2-form Ω is closed and hence symplectic. As a consequence, the Jacobi's identity is satisfied for the corresponding Hamiltonian operator $J_0 = K^{-1}$ obtained by inverting K in (3.4)

$$J_0 = \begin{pmatrix} 0 & \frac{1}{u_{w\bar{w}}} \\ -\frac{1}{u_{w\bar{w}}} & \frac{v_{\bar{w}} - i u_{x\bar{w}}}{2u_{w\bar{w}}^2} D_w + D_w \frac{v_{\bar{w}} - i u_{x\bar{w}}}{2u_{w\bar{w}}^2} + \frac{v_w + i u_{xw}}{2u_{w\bar{w}}^2} D_{\bar{w}} + D_{\bar{w}} \frac{v_w + i u_{xw}}{2u_{w\bar{w}}^2} \end{pmatrix} \quad (3.7)$$

that is explicitly skew-symmetric. The direct proof of the Jacobi's identity for J_0 was also performed with the use of the functional multi-vectors criterion of Olver [7].

The Hamiltonian density was calculated as

$$H_1 = \pi_u u_t + \pi_v v_t - L$$

with the result

$$H_1 = \frac{1}{6} \left[(3v^2 - u_x^2) u_{w\bar{w}} - u_w u_{\bar{w}} u_{xx} + u_x (u_{\bar{w}} u_{xw} + u_w u_{x\bar{w}}) \right] - \varepsilon u. \quad (3.8)$$

CMA system can now be written in the Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \quad (3.9)$$

where H_1 is given in (3.8) and δ_u and δ_v are Euler-Lagrange operators [7] with respect to u and v applied to the Hamiltonian density H_1 (they correspond to variational derivatives of the Hamiltonian functional $\int_V H_1 dV$).

4 Conclusion

We have discovered a symplectic and Hamiltonian structure of the complex Monge-Ampère equation set into a two-component evolutionary form. This is the first step to obtain a multi-Hamiltonian structure of CMA . The next step is to construct a recursion operator for symmetries that, acting on the first Hamiltonian operator, will generate a second Hamiltonian operator and a bi-Hamiltonian representation of the complex Monge-Ampère equation. This work is now in progress and close to final.

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